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Short distance behaviour of the effective string

Michele Caselle
Dipartimento di Fisica Teorica dell'Università di Torino and
I.N.F.N., via P.Giuria 1, I-10125 Torino, Italy
E-mail: caselle@to.infn.it

Martin Hasenbusch
Department of Physics and Astronomy, University of Southampton
Southampton, SO17 1BJ, U.K.
E-mail: hasenbus@phys.soton.ac.uk

Marco Panero
School of Theoretical Physics, Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland
E-mail: panero@stp.dias.ie

Abstract: We study the Polyakov loop correlator in the $(2+1)$ dimensional $\mathbb{Z}_2$ gauge model. An algorithm that we have presented recently, allows us to reach high precision results for a large range of distances and temperatures, giving us the opportunity to test predictions of the effective Nambu-Goto string model. Here we focus on the regime of low temperatures and small distances. In contrast to the high temperature, large distance regime, we find that our numerical results are not well described by the two loop-prediction of the Nambu-Goto model. In addition we compare our data with those for the SU(2) and SU(3) gauge models in $(2+1)$ dimensions obtained by other authors. We generalize the result of Lüscher and Weisz for a boundary term in the interquark potential to the finite temperature case.

Keywords: Confinement, Lattice Gauge Field Theories.

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1. Introduction

Recently a renewed interest has been attracted by the effective string description of the interquark potential in lattice gauge theories (LGT) [1]–[11]. On one hand, in [2, 4] high precision simulations of the interquark potential were run, using algorithms [1, 4, 5] that allow to measure the Polyakov loop correlation function with highly reduced variance. On the other hand, in [9] the lowest excited states in the spectrum of the confining flux tube in the (2+1) dimensional $\mathbb{Z}_2$ gauge model were studied as well, using a variant of earlier methods. These works enabled one to test some longstanding conjectures on the effective string description of the interquark potential, and also triggered some new theoretical effort toward a better understanding of the effective string model itself. While it seems by now

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understood that the first order correction to the interquark potential is given by the so called Lüscher term [12], several other issues are still open and require further investigation. A tentative list of these open problems includes:

The determination of higher order terms (and possibly the full functional form) of the effective string action. The presence of the Lüscher term in the interquark potential only tells us that, at leading order, the effective string action is simply a two-dimensional quantum field theory of $d-2$ free bosonic fields (one for each transverse degree of freedom of the fluctuating string). It does not help to identify the higher order terms in the action, which should describe the string self-interaction. To this end, one has to evaluate higher order corrections (i.e. higher order powers in $1/R$, R being the interquark distance) in the interquark potential. We addressed this problem in two recent papers [3, 4], testing the simplest possible effective action, i.e. the Nambu-Goto string (see below for a detailed discussion) in the large distance, finite temperature regime of the 3d Ising gauge model. We found an impressive agreement between Monte Carlo data and the prediction from the Nambu-Goto action truncated at second order,[1] but apparently no room was left for higher order corrections, which should necessarily be present if the Nambu-Goto string is the correct picture.

The universality issue. As we mentioned above, the first order correction to the interquark potential (i.e. the Lüscher term) shows an impressive degree of universality and has been detected with the precise numerical value predicted by the effective string theory in all of the models studied up to now (ranging from the 3d gauge Ising model to the SU(2) and SU(3) models, both in $d = 3$ and in $d = 4$). Which is the situation for higher order corrections? Do they also show the same universal behaviour? Preliminary results suggest that this is not the case [2], however these evidences were obtained in the short distance regime only (at present only in the case of the 3d Ising model one can reach the large distance regime), where non-universal boundary terms are present (see last item below) and make the analysis much more difficult.

The boundary corrections. As it was recently pointed out by Lüscher and Weisz, the presence of two boundaries (the two Polyakov loops) in the finite temperature geometry makes it necessary to include in the possible higher order terms of the string action also contributions localized at these boundary (which we shall call in the following "boundary terms") which are proportional to a non-universal parameter, which we shall denote as b in the following. These terms induce corrections to the expected form of the interquark potential (which we shall call "boundary corrections" in the following): these corrections are proportional to b. The open problem in this case is to evaluate b in the various LGT's, to check if it can be compatible with zero and more generally if it depends on some relevant feature of the underlying LGT. To this end, it would be very useful to derive the whole functional form (i.e. the explicit dependence from R and L) of the boundary correction.

[1] A similar agreement was observed in [13] where the large distance regime of the interface potential in the 3d Ising model was studied.

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High precision test of the Lüscher correction. While, as we mentioned above, it is by now accepted that the first order correction to the interquark potential is given by the Lüscher term, a high precision test of this result in the finite temperature regime is still missing. It would be interesting to have a quantitative estimate both of the range of values of the interquark distance in which the Lüscher term well describes the interquark potential and of the numerical uncertainty in the determination of its value. This is particularly important in view of the recent results [2, 6] which show that the bosonic string seems to work at surprisingly short distances.

A "Casimir energy paradox"? Finally, we mention the fact that in [9] the authors presented numerical results showing that in a distance range (below 1 fm) where the Lüscher term in the interquark ground state potential is already observed, they only found a few stable excitations of the confining flux, whose spectrum turned out to be grossly distorted compared with the string prediction.

The aim of this paper is to address some of these issues by looking once again at the 3d gauge Ising model, which allows to perform very precise simulations while keeping all of the possible sources of systematic error under control. To this end, we performed a set of new simulations in the *short distance, low temperature* regime of the interquark potential, which we used first to make a high precision test of the Lüscher term (as discussed above) and then to study the higher order terms of the effective string action. In this respect the present paper can be considered as the completion of our previous paper [4] in which we performed a similar study in the *large distance, high temperature* regime of the interquark potential. Our results can also be compared with the findings of an analogous study, recently published in [9]. A preliminary account of our results can be found in [14]. A non-trivial problem one has to face when looking at the higher order effective string terms is the presence of the non-universal boundary correction discussed above. In order to deal with this problem we evaluated the functional form of the boundary correction (a result which is rather interesting in itself as we shall see below), in the framework of the zeta-function regularization. In this way, we may predict the large distance behaviour of the boundary correction, and use the high precision results of [4] to fix the non-universal constant in front of it. As we shall see, our data strongly suggest that this value is compatible with zero for the 3d gauge Ising model.

A side consequence of this analysis is that we shall be able to show in a rather unambiguous way that, at least in the short distance regime the interquark potential, in the 3d Ising model cannot be described by a simple Nambu-Goto string. We shall then conclude the paper with a tentative comparison of our results with the short distance behaviour of other LGT's trying to shed some light on the universality issue discussed above.

This paper is organized as follows. We start in section 2 with a short summary of the effective string description of the interquark potential. Then in the following two sections we compare our new Monte Carlo results with the effective string predictions in the short distance regime. In section 3, we discuss the first order correction (the "Lüscher" term), while in section 4 we address the issue of the higher order corrections. Section 5 is then devoted to the study of the functional form of the boundary correction. These results are

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used in section 6 to study the boundary correction in the large distance regime. Section 7 is devoted to some concluding remark and to the comparison of our Ising results with those obtained in the SU(2) and SU(3) gauge theories in $d = 3$. Finally, the detailed derivation of the functional form used in section 5 is reported in the appendix.

2. Summary of known theoretical results on the effective string description of the interquark potential

We refer to our previous papers [3, 4] for a detailed discussion about the effective string picture and its peculiar realization in the finite temperature geometry (i.e. in the case of Polyakov loop correlators). Here, we only recall some basic facts (more details can be found in the appendix).

- In finite temperature LGT's the interquark potential can be extracted by looking at correlators of Polyakov loops in the confined phase. The correlator of two loops $P(x)$ at a distance R and at a temperature $T = 1/L$ (L being the size of the lattice in the compactified "time" direction) is given by

$$G(R) \equiv \langle P(x)P^{\dagger}(x+R)\rangle \equiv \mathrm{e}^{-F(R,L)} \,. \tag{2.1}$$

  The free energy $F(R, L)$ is expected to be described, as a first approximation, by the so called "area law"

$$F(R,L) \sim \sigma LR + k(L) \,, \tag{2.2}$$

  where σ denotes the string tension and $k(L)$ is a non-universal constant depending only on L.

- According to the effective string picture, in the rough phase of the theory one must add to the area law of eq. (2.2) a correction due to quantum fluctuations of the flux tube ("effective string corrections"). Such a correction is expected to be a complicated function of R and L, but if one neglects the contributions due to the string self-interaction terms and from the boundary terms which we shall discuss below, then one finds

$$F(R,L) \sim F_q(R,L) = \sigma LR + k(L) + F_q^1(R,L) \tag{2.3}$$

  with

$$F_q^1(R,L) = (d-2)\log\left(\eta(\tau)\right) ; \qquad -i\tau = \frac{L}{2R} \,, \tag{2.4}$$

  where $(d-2)$ is the number of transverse dimensions, η denotes the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) ; \qquad q = e^{2\pi i\tau} \,, \tag{2.5}$$

  and the labels q and 1 in F_q^1 recall that this is the *first order* term in the expansion of the *quantum* fluctuations of the flux tube. Eq. (2.3) is referred to as the "free string approximation".

- The Dedekind function has two different expansions — both of which turn out to be convergent for any finite value of L and R — which are most suitable for the two regimes $2R < L$ and $2R > L$, respectively. These expansions are related to each other by the so called "modular transformation".

  For $2R < L$

$$F_q^1(R,L) = \left[-\frac{\pi L}{24R} + \sum_{n=1}^{\infty} \log(1 - e^{-\pi n L/R}) \right] (d-2) \,. \tag{2.6}$$

  The first term of this expansion is the well known "Lüscher term" [12].

  For $2R > L$

$$F_q^1(R,L) = \left[-\frac{\pi R}{6L} + \frac{1}{2} \log \frac{2R}{L} + \sum_{n=1}^{\infty} \log(1 - e^{-4\pi n R/L}) \right] (d-2) \,. \tag{2.7}$$

  In this case, the first term of the expansion is proportional to R and acts to lower the string tension.

  Unless we are in the intermediate region $R \sim L/2$, the exponentially decreasing terms which appear in eq.s (2.6) and (2.7) can be neglected. We shall largely use this approximation in the following.

- At large enough temperatures, i.e. for small values of L, or small enough values of R, higher order terms in the expansion of the flux tube quantum fluctuations become important and cannot be neglected. These terms encode the string self-interaction and depend on the particular choice of the effective string action. The simplest proposal (discussed in [3, 4]) is the Nambu-Goto action, in which the string configurations are simply weighted by the area of the world-sheet. Its contribution to the free energy turns out to be (fixing for simplicity $d = 3$) (see [15] and [4])

$$F_q^2(R,L) = -\frac{\pi^2 L}{1152\,\sigma R^3} \left[2E_4(\tau) - E_2^2(\tau) \right] , \tag{2.8}$$

  where E_2 and E_4 are the Eisenstein functions. The latter can be expressed in power series

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \tag{2.9}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \tag{2.10}$$

$$q \equiv e^{2\pi i \tau} \,, \tag{2.11}$$

  where $\sigma(n)$ and $\sigma_3(n)$ are, respectively, the sum of all divisors of n (including 1 and n), and the sum of their cubes.

β	$L_c \equiv 1/T_c$	L	N_s	ξ	σ	R_c
0.65608	2	20	32	0.644(1)	0.20487(1)	2.71
0.73107	4	20,40,80	64	1.456(3)	0.0440(3)	5.84
0.75180	8	80	128	3.09(1)	0.010560(18)	11.92

Table 1: A few information on our simulations. In the first column the value of β is given and in the second the inverse of the critical temperature. The third and fourth columns contain the values of lattice size in the "temperature" direction L and in the two spacelike directions N_s. In the last three columns the values of the exponential correlation length (obtained by interpolating the values reported in [17] and [18] and by the analysis of the low temperature series [19]), the value of the string tension in dimensionless units, and the value of R_c defined as $R_c \equiv \sqrt{1.5/\sigma}$ (see section 2.7 of ref. [4]) are provided.

3. High precision test of the Lüscher term

3.1 The simulations

In order to perform a high precision test of the Lüscher term we run a set of Monte Carlo simulations in the short distance regime. In table 1 we report some information on our simulations. The choice of very large lattice sizes in the "temperature" direction (i.e. the direction of the Polyakov loops) ensures us that we are in the very low temperature domain. The temperature of our present simulations ranges from $T/T_c = 1/20$ for $L = 80$ at $\beta = 0.73107$ up to $T/T_c = 1/5$ for $L = 20$ at $\beta = 0.73107$. The values for the $1/L$ corrections estimated in [4] indicate that for $T \leq T_c/10$ the contribution of possible higher order effective string terms due to the finiteness of the lattice size in the temperature direction is completely negligible within the precision of our data while the data at $T = T_c/5$ are at the border of our precision. Thus, at least for $T \leq T_c/10$ we may neglect possible $1/L$ corrections and we can concentrate on the $1/R$ corrections only. Notice that this is exactly the opposite situation with respect to our previous paper [4] in which the temperature was much higher and the $1/L$ corrections were dominating. The sample at the smallest value of β: $\beta = 0.65608$ is characterized by a rather small correlation length and we should therefore expect large scaling violations.

For a detailed discussion of the algorithm that we used, of the simulation setting and more generally on the 3d gauge Ising model, we refer the reader to our previous paper [4]. The results of our simulations are reported in table 2 and in table 3.[2]

3.2 Test of the Lüscher term

We performed our analysis in three steps. First we made, following what is usually done in the literature, a rather naive test by directly fitting the data with a $1/R$ correction. As we

[2]Notice that also the result at $R = 80$ listed in table 3 and those at $R = 40$ listed in table 2 are meaningful (even if $80 > 128/2$ and $40 > 64/2$) due to the non-trivial mapping of the boundary conditions from the Ising gauge to the Ising spin model under duality transformation (see [16], section 4.3 for a detailed discussion of this point). In particular, performing our simulations in the 3d Ising spin model (as we did), we neglected the anti-periodic boundary conditions, which would produce the "echo" contribution due to the distances (128-80) and (64-40) respectively.

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R	$L = 20$	$L = 40$	$L = 80$
4	0.370163(58)	0.136976(25)	0.0187711(45)
5	0.382971(60)	0.146536(27)	0.0214749(53)
6	0.391093(62)	0.152723(31)	0.0233307(56)
7	0.396557(66)	0.157044(31)	0.0246517(61)
8	0.400670(68)	0.160041(33)	0.0256028(68)
9	0.403491(72)	0.162210(33)	0.0263272(71)
10	0.405866(72)	0.163930(36)	0.0268669(76)
11	0.407682(72)	0.165229(36)	0.0272785(81)
12	0.409213(77)	0.166123(36)	0.0276178(83)
13	0.410384(82)	0.166999(38)	0.0278800(84)
14	0.411507(81)	0.167647(38)	0.0280970(86)
15	0.412345(87)	0.168165(41)	0.0282665(88)
16	0.413234(86)	0.168697(41)	0.0284264(93)
18	0.414431(90)	0.169435(43)	0.0286463(95)
20	0.415658(95)	0.169883(43)	0.0288094(97)
30	0.418863(110)	0.171406(48)	0.0292359(109)
40	0.420810(124)	0.171981(50)	0.0293721(109)

Table 2: Values of the ratio between two successive Polyakov loop correlators $G(R+1)/G(R)$ for various values of R and L, at $\beta = 0.73107$.

shall see below, this choice is correct only in a very narrow range of interquark distances, since for large enough values of R the exponentially decreasing terms $\exp(-\pi n L/R)$ contained in eq. (2.6) start to matter and cannot be neglected. In the second level of analysis we fit the data with the whole functional form of the free bosonic correction, eq. (2.6), finding a remarkable agreement between the data and our theoretical expectations. Finally in the third level of analysis we combine the data in such a way that the string tension is eliminated as parameter. Thus, we can make an absolute comparison, with no free parameter to fit, between our data and the effective string prediction. Before going into the details of the analysis let us stress a point which will be important in the following. Due to the algorithm we used, each number in tables 2 and 3 corresponds to a different simulation. Hence the sets of data we fit are completely uncorrelated and we can safely trust both on the χ^2 obtained from the fit and on the best fit results for the parameters.

3.2.1 Fitting with a $1/R$ correction

We fitted the data reported in table 2 and table 3 according to the law

$$-\frac{1}{L}\log\left(\frac{G(R+1)}{G(R)}\right) = a - b\left(\frac{1}{R+1} - \frac{1}{R}\right). \tag{3.1}$$

With this normalization the bosonic string model predicts $a = \sigma$ and $b = \pi/24 = 0.13089\ldots$

If we fit, for a given choice of β and L, all the values of R listed in the tables, the reduced χ^2 turns out to be very high and the best fit values of a and b are very far from

R	$L = 80$
8	0.382613(58)
10	0.396313(61)
12	0.405088(65)
14	0.411017(69)
16	0.415268(71)
18	0.418048(74)
20	0.420344(74)
22	0.422062(78)
24	0.423323(78)
26	0.424486(78)
28	0.425496(81)
30	0.425958(81)
32	0.426747(82)
36	0.427513(83)
40	0.428255(84)
60	0.430097(87)
80	0.431068(88)

Table 3: Same as table 2 but for the data at $\beta = 0.75180$.

R	$L = 20$
2	0.010583(3)
3	0.013032(4)
4	0.014370(4)
5	0.015101(4)
6	0.015540(4)
7	0.015817(4)
8	0.016002(4)
9	0.016151(5)
10	0.016240(5)
12	0.016383(5)
14	0.016476(5)
16	0.016549(5)
18	0.016608(5)
20	0.016643(6)

Table 4: Same as table 2 but for the data at $\beta = 0.65608$.

the expected values. This indicates that at short R we have higher order (string) corrections which are proportional to higher powers of $1/R$ and compromise the fit. The standard way to deal with this type of behaviour is to repeat the fit eliminating the data one after the other starting from those with the lowest values of R, until for some value of $R_{\rm min}$ an acceptable (i.e. order unity) reduced χ^2 is finally reached. However it turns out that for all our samples, except the one at $T = T_c/20$ this scenario never occurs. Instead we see that after a rapid decrease, the χ_r^2 starts again to increase as $R_{\rm min}$ is increased and the minimum of this shape never goes below 1. As an example, we report in table 5 the results of the fit for the data at $\beta = 0.73107$ and $L = 40$. The reason for this behaviour is that corrections for *large* values of R are present in the data. These corrections are nothing but the exponentially decreasing terms of eq. (2.6). If we neglect them in the fit, we shall never be able to extract the correct value of the Lüscher term, as the best fit values reported in table 5 clearly show. If the temperature is low enough, there is a range of values of R in which both the small R and the large R corrections become negligible and the expected value of the Lüscher term can be recovered. This range can be found by iteratively eliminating the smallest and the largest values of R in the fit and *looking for a stable and acceptable value of* χ_r^2.

However, if T is not small enough, this procedure does not converge. This is the case for instance of our data at $\beta = 0.73107$ and $L = 20$. We report in table 6 the best fit results obtained in this way for the other three samples.

$R_{\rm min}$	χ^2_r	σ	b
4	300	0.044123(2)	0.1147(1)
5	108	0.044067(2)	0.1211(2)
6	48	0.044033(2)	0.1260(2)
7	31	0.044012(3)	0.1295(3)
8	21	0.043995(3)	0.1329(4)
9	15	0.043981(4)	0.1361(6)
10	13.2	0.043970(4)	0.1390(7)
11	9.9	0.043958(4)	0.1426(9)
12	5.4	0.043945(5)	0.1470(11)
13	5.1	0.043939(5)	0.1492(14)
14	3.9	0.043931(6)	0.1525(17)
15	3.4	0.043925(6)	0.1555(21)
16	4.2	0.043924(7)	0.1570(26)

Table 5: Results of the fit according to eq. (3.1) for the data at $\beta = 0.73107$ and $L = 40$. In the first column the minimum value of R included in the fit is given. The second column contains the reduced χ^2. In the last two columns the best fit results for σ and for the Lüscher term are given, respectively.

β	L	T/T_c	R	χ^2_r	σ	b
0.73107	80	1/20	10-20	0.6	0.044030(4)	0.1304(6)
0.73107	40	1/10	9-13	1.5	0.044042(10)	0.1285(12)
0.75180	80	1/10	18-26	0.4	0.010532(4)	0.1264(19)

Table 6: Results of the fit according to eq. (3.1). In the first three columns the values of β and L of the data used in the fit and the corresponding value of T/T_c are provided. In the fourth column, we give the range of values of R included in the fit. In the fifth column the reduced χ^2 is given. The last two columns contain the best fit results for σ and for the Lüscher term.

We insisted in this type of fit because it is widely used in the literature and, as we have seen, it can easily lead to misleading conclusions. In particular one should be aware that:

- if L is not large enough, i.e. if the temperature is not small enough, the range of interquark distances within which this type of fit can be performed vanishes. The precise threshold in temperature depends on the precision of the data. In our case the threshold was around $T \sim T_c/10$.

- If one nevertheless performs the fit outside the allowed region the reduced χ^2 becomes larger than 1 and the values of the Lüscher term which one extracts from the fit turns out to be (erroneously) different from the expected $\pi/24$ factor.

- If one enforces the expected $\pi/24$ coefficient and performs the fit outside the allowed range then the values of the string tension extracted from the fit may show relevant systematic deviations from the correct value.

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We think that these observations should be carefully taken into account when dealing with the interquark potential extracted from Polyakov loops in any LGT, i.e. also in the physically more interesting SU(2) and SU(3) cases, if the precision reached by the simulations is high enough.

3.2.2 Fitting with the whole functional form of the bosonic string correction

In order to keep into account the next to leading terms in eq. (2.6) we fitted the data reported in table 2 and table 3 according to the law

$$-\frac{1}{L}\log\left(\frac{G(R+1)}{G(R)}\right) = a + \frac{c}{L}\left(F_q^1(R+1,L) - F_q^1(R,L)\right). \tag{3.2}$$

With this normalization a is again the string tension while $c = 1$ means that we have a perfect agreement between the whole bosonic string prediction (Lüscher term plus exponentially decreasing corrections) and the data. In table 7 we report as an example the result of the fits in the particular case $\beta = 0.75180$, $L = 80$. Similar results are obtained with the other values of β and L.

We find a very good agreement between our data and the bosonic effective string prediction, *keeping in the fit all the interquark distances up to* $R = 80$. The uncertainty in our determination of c is less than 1%. The decrease in the reduced χ^2 indicates that the higher order string contributions can be neglected for $R \geq 20$. Looking at the string tension results, we see that the best fit value changes by a rather large amount (with respect to the quoted errors) as R_{min} increases. This observation has two relevant implications:

- As we mentioned above, a large systematic deviation in the best fit estimates of the string tension has to be expected, if the effective string corrections are not properly taken into account.

R_{min}	χ^2_r	σ	c
8	181	0.0105622(7)	0.824(1)
10	47	0.0105454(8)	0.883(2)
12	15	0.0105364(9)	0.922(2)
14	6.6	0.0105315(10)	0.949(3)
16	3.9	0.0105285(11)	0.967(5)
18	1.2	0.0105255(12)	0.990(6)
20	1.1	0.0105247(14)	0.996(8)
22	1.0	0.0105241(15)	1.003(10)

Table 7: Results of the fit according to eq. (3.2) for the data at $\beta = 0.75180$, $L = 80$. The first column contains the minimum value of R included in the fit (recall that for this value of β the correlation length is twice the one of $\beta = 0.73107$ and a similar doubling of all the length scales must be kept into account when comparing the data at the two values of β). In the second column the reduced χ^2 is given. In the last two columns we provide the best fit results for σ and for the coefficient in front of the bosonic string correction.

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β	L	T/T_c	R_{min}	χ^2_r	σ	c
0.65608	20	1/10	9	0.58	0.204864(9)	1.017(11)
0.73107	80	1/20	10	1.3	0.044023(3)	1.003(4)
0.73107	40	1/10	10	2.0	0.044019(4)	1.005(5)
0.73107	20	1/5	10	0.67	0.043985(4)	0.988(6)
0.75180	80	1/10	22	1.0	0.0105241(15)	1.003(10)

Table 8: Results of the fit according to eq. (3.2). In the first three column the values of β and L of the data used in the fit and the corresponding value of T/T_c are summarized. In the fourth column the minimum value of R included in the fit is quoted. In the fifth column we give the reduced χ^2. The last two columns contain the best fit results for σ and for the coefficient in front of the effective string correction.

- Even if the first order effective string correction is properly taken into account, systematic deviations in the string tension may still appear if too small values of R are included in the fits.

Our results for the other values of β are summarized in table 8. Looking at this table one can see that the data do not show scaling violations and that the expected value $c = 1$ is found even for $\beta = 0.65608$ (for which instead any bulk observable shows very large scaling violations) ! This represents a further nontrivial test of the whole picture, since the effective string picture is indeed expected to hold in the whole range $\beta_r < \beta < \beta_c$ (β_r and β_c being respectively the roughening and the deconfinement critical points of the gauge model) and should abruptly break down for $\beta < \beta_r$. It is important to recall that in the above fits we neglected the short distance data. As we shall see below the short distance deviations from the free string behaviour are instead affected by scaling corrections.

In the following section we shall make a further step in our analysis. We shall perform some kind of "absolute test" of the effective string, combining the data so as to eliminate the string tension from the game as well, thus only leaving the effective string corrections in the data.

3.2.3 An "absolute" test of the effective string picture

This can be easily done by building the combinations

$$H(R_1, R_2) \equiv -\frac{1}{L} \log \left(\frac{G(R_1+1)G(R_2)}{G(R_1)G(R_2+1)} \right) \tag{3.3}$$

in which the string tension disappears. We plot in figure 1 and 2 a sample of the results we obtain. In particular we concentrate on the sample at the largest value of β and fix one of the two entries to the first acceptable value of R_{min} as extracted from the previous analysis, i.e. $R_2 = 20$. In figures 1 and 2 we compare our results with the bosonic string prediction for $H(R, 20)$ obtained using eq. (2.6) and also with what one would obtain (dashed lines in figures 1 and 2) keeping only the $1/R$ correction in eq. (2.6). We find for $R_1 > 20$ an impressive agreement between our data and the effective string predictions. All

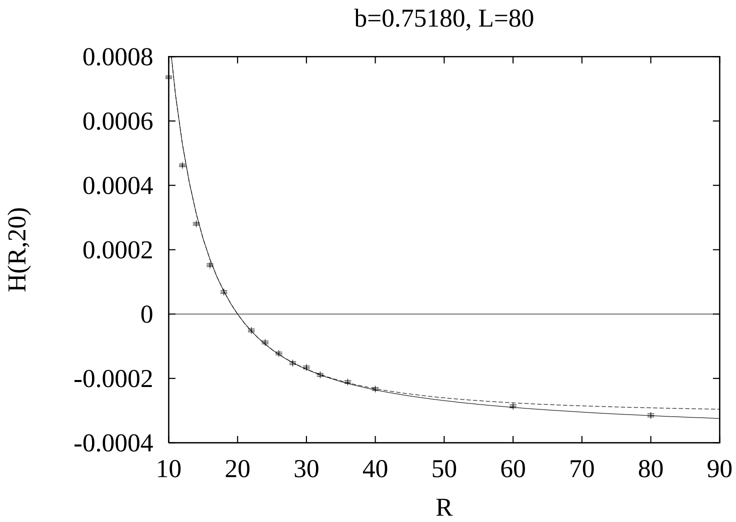

Figure 1: Comparison between our values of $H(R, 20)$, the bosonic string prediction, eq. (2.6) (continuous line) and the $1/R$ approximation to eq. (2.6) (dashed line) for the data at $\beta = 0.75180$, $L = 80$.

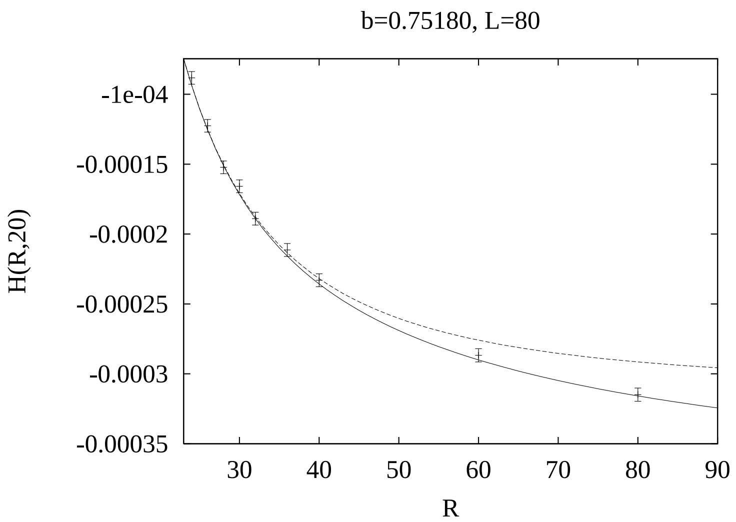

Figure 2: Same as figure 1, but with a larger resolution.

the data agree within the errors. Instead, for $R_1 < 20$ the large deviations from the free effective string appear. We shall discuss these deviations in the forthcoming sections. It is very interesting to observe that, in agreement with the results of the first level analysis discussed above, for large enough values of R the contribution due to the next to leading, exponentially decreasing, terms of eq. (2.6) is clearly visible in the data. It is clear that these terms cannot be neglected in the fits.

Finally in order to have a global view of all our data we built the differences

$$\Delta(R_1, R_2) \equiv H(R_1, R_2) - \frac{1}{L}\left(F_q^1(R_1+1, L) + F_q^1(R_2, L) - F_q^1(R_2+1, L) - F_q^1(R_1, L)\right). \tag{3.4}$$

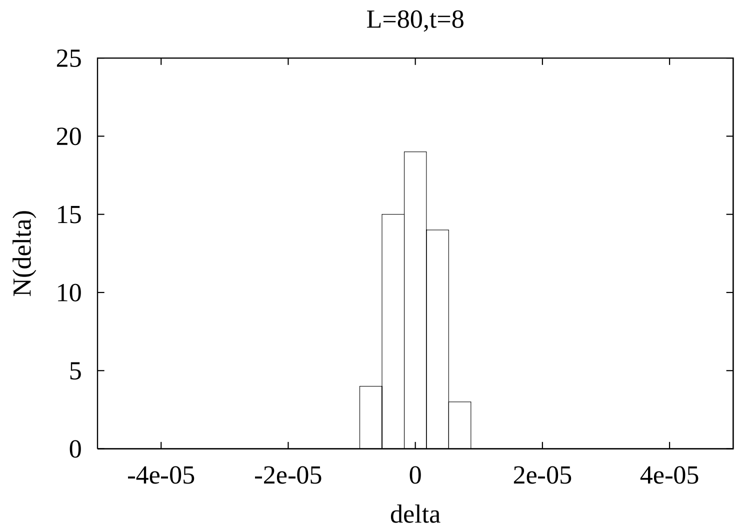

Figure 3: Histogram of $\Delta(R_1, R_2)$ for all the values of $R_1, R_2 \geq 20$. Notice the very small scale of the Δ axis.

With this definition, we have $\Delta = 0$ for all choices of R_1 and R_2 if the free effective string picture of eq. (2.6) is the correct description of the data. We report in figure 3 as an example of the results that we obtained the histogram of Δ for $\beta = 0.75180$, $L = 80$ for all the values of $R_1, R_2 \geq 20$. The overall agreement between the data and the effective string prediction is very good. With the only exception of the $L = 20$ set the data show a rather nice gaussian like distribution around the expected $\Delta = 0$ value. The variance of these gaussians is of the same order of the errors of $H(R_1, R_2)$ extrapolated from those reported in table 2 and 3 and no systematic deviation or asymmetry is visible in the data. These observations suggest that the (very small) spread around the $\Delta = 0$ value is a purely statistical effect (as a side remark this is also a test of the fact that our statistical errors are reliable). On the contrary the sample at $L = 20$ shows some systematic asymmetry. This is an indication that for $T \geq T_c/5$ the large distance, high temperature effects discussed in [4] start to give non-negligible contributions within the precision of our data.

3.2.4 The zero temperature string tension

As a byproduct of the previous analysis we may obtain from our fits a very precise determination of the zero temperature string tension for three sets of data. They are reported in table 8 where we have listed the results of the fits according to eq. (3.2).

The values of σ that we find for $\beta = 0.75180$ is slightly smaller than the value we used in our previous papers [3, 4], which was $\sigma = 0.010560(18)$. The effect is very small, but nevertheless it is outside the quoted error bars (while in the case of $\beta = 0.73107$ the error bars are too large to detect this effect). An interesting consequence of this (very small) rescaling of σ is that it could explain the residual (small) systematic deviation we found in [4] between our data and the Nambu-Goto effective string truncated at the first perturbative order (see figures 2, 3, 4 and 5 of [4]). Both the sign of these deviations and the fact that they are proportional to L are in agreement with this explanation.

4. Higher order corrections at short distance

With the above results at hand, we are in the position to address the issue of the higher order correction at short distance in a reliable and precise way. We shall perform this analysis in two steps. First we shall study the samples at $\beta = 0.73107$ and $\beta = 0.75180$ trying to quantify the higher order terms in the potential, without referring to any particular effective string picture. Then, in the second step we shall compare our results with the expected behaviour of the Nambu-Goto string. In this second step we shall also include the data at $\beta = 0.65608$ in our analysis.

4.1 Higher order corrections

We decided to study the higher order corrections by looking at the observable (see the def. of eq. (3.4)) $\Delta(R, R-n)$.

This quantity is interesting for several reasons. First of all, as we mentioned above, it is different from zero only if higher order corrections are present. Moreover, for $n = 1$ we may easily relate it with the observable $c(\tilde{r})$ introduced in [2]. The relation is

$$H(R, R-1) = 2\frac{c(\tilde{r})}{\tilde{r}^3} \tag{4.1}$$

(for a definition of $\tilde{r}$ see [2]). As a side remark, this means that $H(R, R-1)$ (suitably normalized) is an estimator of the central charge of the underlying conformal field theory (CFT), (see [4] for a discussion of the CFT description of effective string theory). Thus $\Delta(R, R-1)$ measures the deviation of the central charge of this CFT from the free bosonic value (i.e. $c = 1$).

In order to compare values of this quantity for samples at different values of σ and for different choices of n it is useful to define the following scale invariant version of Δ

$$D\left(R + \frac{n+1}{2}\right) \equiv \frac{\Delta(R+n, R)}{n\sigma^{3/2}}. \tag{4.2}$$

In building this observable we used the high precision results for σ obtained in the previous section.

We are primarily interested in identifying the exponent of the correction, thus we fitted $D(R)$ with the ansatz

$$D(R) = \frac{g_{\alpha+2}}{R^{\alpha+2}}. \tag{4.3}$$

It is possible to relate $g_{\alpha+2}$ to a corresponding higher order correction in the free energy. More precisely, if the free energy has a correction of the type

$$-\log G(R) \simeq \sigma RL + k(L) + \log\left[\eta\left(i\frac{L}{2R}\right)\right] + \frac{\gamma_\alpha}{\sigma^{\frac{\alpha-1}{2}}}\frac{L}{R^\alpha}, \tag{4.4}$$

then in $D(R)$ we expect to find a correction proportional to $g_{\alpha+2}/R^{\alpha+2}$ (plus higher order terms due to the lattice discretization) with $g_{\alpha+2} = \alpha(\alpha+1)\gamma_\alpha$. Notice that in eq. (4.4) we have rescaled the coefficient of the higher order correction with a suitable power of σ so as to

α	χ_r^2	$g_{\alpha+2}$	γ_α
1	2.8	0.033(3)	0.0166(13)
2	1.8	0.053(4)	0.0089(7)
3	1.5	0.079(6)	0.0066(5)
4	1.5	0.114(9)	0.0057(4)

Table 9: Results of the fits according to eq. (4.3) for the data at $\beta = 0.75180$ and $L = 80$. In the first column we give the value of α, in the second column the reduced χ^2, in the last two columns the values of $g_{\alpha+2}$ and γ_α.

make it adimensional. We also implicitly assumed that the higher order term is proportional to L. We shall see below that this assumption agrees with the numerical results.

Looking at the discussion of section 2, it is easy then to relate the presence of a nonzero γ_α term to a precise contribution in the string action. In particular, if the fit with $\alpha = 1$ gave an acceptable (i.e. order unity) reduced χ^2 and a value $\gamma_1 \neq 0$, then this would indicate that the Lüscher term does not properly describe the data and that a different coefficient in front of the $1/R$ correction to the interquark potential is needed in this short distance regime. If instead the $\alpha = 2$ fit gave an acceptable χ_r^2, then this would indicate that a boundary term might be present in the data, and we could estimate the boundary parameter b (we shall discuss the boundary contribution in section 5 below). Notice that, according to our normalization choice, the parameter b introduced by Lüscher and Weisz in [2] is given by $b = -\frac{24}{\pi}\frac{\gamma_2}{\sqrt{(\sigma)}}$. We shall discuss this issue in detail below. Finally, an acceptable χ_r^2 with $\alpha = 3$ would indicate that higher order terms are present in the effective string action. In particular, the Nambu-Goto action would precisely give such a contribution (see below). The value of γ_3 extracted from the fit could help to identify the form of these higher order terms. Clearly, once for a given α we find $\gamma_\alpha \neq 0$ all the higher powers of R in the fit would have nonzero coefficients, regardless of the presence of a corresponding term in the action, simply because of the lattice artifacts. It is well possible that a mixture of all these terms is present in the action, but unfortunately our data are not precise enough to allow for more than one parameter fits. Thus we shall only be able to identify (if it exists) the leading term among them. As a consequence one should look at the results which we shall now discuss more as a qualitative indication than a quantitative estimate of the higher order terms in the string action. Notwithstanding this, a few precise pieces of information can be obtained from our data.

As a last point let us mention that it is important to properly select the range of values of R to be included in the fit. According to [3] and [4], we must choose $R \geq R_c$ (see the values reported in table 1).

We performed the fit for various integer values of α; the results are listed in tables 9–11 (see also figure 4 and figure 5) for all the data except those at $L = 20$ (which, as we have shown in the previous section, seem to be affected by high temperature corrections).

Let us briefly comment on these results.

1. The three sets of data show a remarkably similar pattern.

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α	χ^2_r	$g_{\alpha+2}$	γ_α
1	22.1	0.037(1)	0.0182(4)
2	6.8	0.052(1)	0.0087(2)
3	1.5	0.070(1)	0.0059(1)
4	1.0	0.092(2)	0.0046(1)

Table 10: Same as table 9 but for the data at $\beta = 0.73107$ and $L = 80$.

α	χ^2_r	$g_{\alpha+2}$	γ_α
1	10.4	0.036(1)	0.0180(6)
2	4.1	0.052(2)	0.0087(3)
3	1.8	0.070(2)	0.0058(2)
4	1.4	0.092(3)	0.0046(1)

Table 11: Same as table 9 but for the data at $\beta = 0.73107$ and $L = 40$.

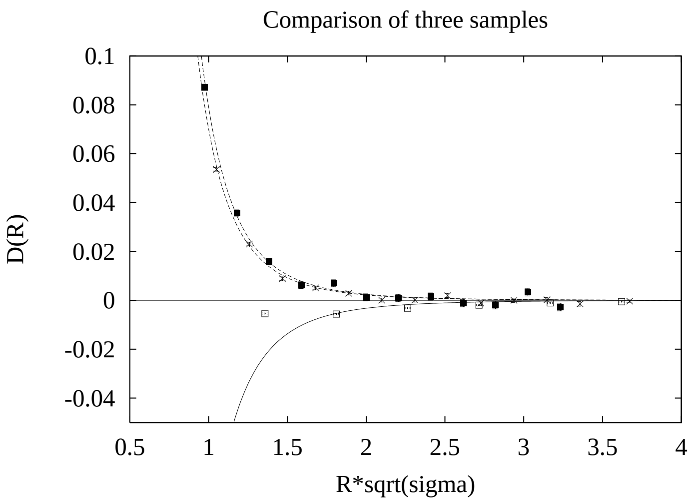

Figure 4: Values of $D(R)$ versus $R\sqrt{\sigma}$ for the sample at $\beta = 0.65608$ (white squares), the one at $\beta = 0.75180$ (black squares) and the $L = 40$ one at $\beta = 0.73107$ (crosses). The continuous line is the Nambu-Goto expectation, the two dashed lines are the best fit results for $\beta = 0.73107$ and $\beta = 0.75180$ discussed in the text.

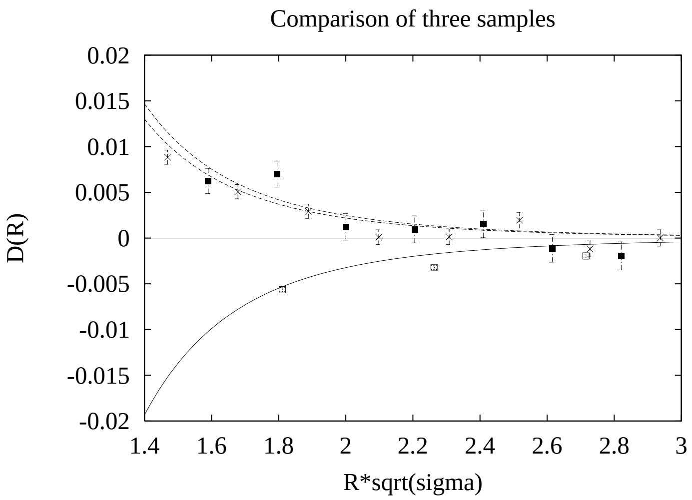

Figure 5: Same as figure 4 but with a higher resolution.

2. Looking at the χ^2 values it is clear that our data exclude a $\alpha = 1$ correction. This means that the Lüscher term we have subtracted perfectly describes the $1/R$ behaviour of the data.

3. A boundary term ($\alpha = 2$) gives reduced χ^2's which are higher than the $\alpha > 2$ ones in all three cases. If we assume that the data are described by a boundary correction, then the amplitude of such a correction turns out to be of the same order of magnitude (and sign) of those that we find in the large distance regime (see the discussion at the end of section 5), but the scaling behaviour of the large distance results is not the one expected for a boundary term.

4. The fits seem to support a value between 3 and 4 for α. This result is compatible with a $1/R^3$ "Nambu-Goto like" correction or, more likely, a mixture of a $1/R^3$ plus higher order corrections.

5. The two sets of results at $\beta = 0.73107$ are compatible within the errors. This means that also the higher order corrections in the free energy $F(R, L)$ scale linearly with L.

6. Comparing the sets of data at $\beta = 0.73107$ and the one at $\beta = 0.75180$ we see a good scaling behaviour in β. This is also evident if one looks at figure 4 in which the two sets of data are plotted together.

4.2 Comparison with the Nambu-Goto string

The most impressive feature of the above analysis is that, although the power that we find $1/R^3$ is exactly the one expected according to the Nambu-Goto action, the value of the coefficient $\gamma_3 \sim 0.006$ that we find is definitely different with respect to the Nambu-Goto prediction eq. (2.8), namely $\gamma_3 = -\pi^2/1152 = -0.00856$, which is opposite in sign and more or less 4/3 in magnitude with respect to our fit's results.[3]

In figures 4 and 5 we plot our data for the three values of β (for $\beta = 0.73107$ we chose the sample with $L = 40$). For comparison, we also plot the Nambu-Goto expected behaviour (solid line) and the two curves obtained using the two best fit values $\gamma = 0.0059$ and $\gamma = 0.0066$ for $\beta = 0.73107$ and $\beta = 0.75180$ respectively (dashed lines).

Looking at these figures, we see that the data for $\beta = 0.65608$ show a large scaling violation. This had to be expected due to the rather small value of the correlation length $\xi < 1$ for this β. Note that even the free energy of a free field theory on a lattice shows corrections of the order $1/R^3$. As a little exercise, we have computed the free energy of the free field theory on a square lattice with the appropriate boundary conditions. We have used the most naive discretisation of the derivate of the field. We would expect that the artifacts in the Ising model are at least as large as those in the free field theory on the lattice. Indeed we find that the corrections in the free field theory have the same sign as those of the Ising gauge model at $\beta = 0.65608$, but a smaller amplitude.

[3] A similar disagreement in the short distance regime was recently reported in [9].

On the other hand, the fact that the deviations between $\beta = 0.73107$ and $\beta = 0.75180$ are much smaller than those with $\beta = 0.65608$, makes us confident that for $\beta > 0.73$ lattice artifacts have only a minor impact on the higher order corrections and hence we can extract continuum physics results.

A possible interpretation of this scenario, keeping also into account the results of our previous paper [4] is that, while the Nambu-Goto effective string provides a correct description of finite temperature effects of the interquark potential at large distances, while it fails to describe corrections at small distance R. Along with Dirichlet boundary conditions, some irrelevant operator becomes important and drastically modifies the shape of the interquark potential (or better: the shape of its deviation from the free string behaviour).

The simplest candidate for such a non-universal behaviour is a boundary term of the type discussed in the next section. A positive and large enough value of b could perhaps justify the observed behaviour of $D(R)$. However as we shall see in the next section the large distance behaviour of the interquark potential in the Ising case seems to suggest that $b \sim 0$. Thus it is possible that some other operator, maybe an extrinsic curvature contribution [20, 21] is actually responsible for the observed behaviour. In order to better understand this point we shall devote the following two sections to a detailed discussion of the boundary term.

An alternative scenario is that the short distance breakdown of the Nambu-Goto string description has to be explained by effects beyond an effective string theory; for a discussion we refer the reader to ref. [9].

5. The boundary term

As we have seen in the previous section, in order to understand the short distance behaviour of the effective string potential it would be important to have an independent estimate of the boundary parameter b. This can be achieved by studying the large distance data. However, in order to perform this analysis, we must know the functional form of the boundary correction in the large distance regime. This section is devoted to such calculation, while in the next section we shall compare this result with the large distance data of ref. [4].

5.1 The functional form of the boundary correction

As mentioned in the introduction, the most general effective string action in presence of boundaries requires the inclusion in the action of terms that are localized at the boundary. The simplest possible term of this type is [2]

$$\mathcal{A}_b = \frac{b}{4} \int_0^L dt \left[\left(\frac{\partial h}{\partial z} \right)^2_{z=0} + \left(\frac{\partial h}{\partial z} \right)^2_{z=R} \right], \tag{5.1}$$

where b is a parameter with dimensions [length], h denotes the transverse displacement of the string (see the appendix for a detailed discussion) and the $\frac{1}{4}$ factor has been added to agree with the conventions of [2].

This additional term can be treated in the framework of the zeta-function regularization in a way similar to the Nambu-Goto one. We report in the appendix a detailed account of this calculation and only list here the final result, which turns out to be rather simple.

At first perturbative order in b, the regularization of the free string action plus the boundary term gives the same result as the pure free string action, namely the Dedekind function discussed above, provided one replaces the interquark distance R with

$$R \to R^* = \frac{R}{(1+2\frac{b}{R})^{\frac{1}{2}}} \tag{5.2}$$

(remember that b has the dimension of a length).

Thus, if we denote the effective string contribution to the interquark potential at this order (with an obvious choice of notation) as $F_q^1(R,L) + F_q^b(R,L)$, we find (we shall keep from now on $d = 3$ to simplify the equations)

$$F_q^1(R,L) + F_q^b(R,L) = \log \eta \left(i \frac{L}{2R^*} \right). \tag{5.3}$$

Expanding this result in the short distance regime (i.e. for $2R^* < L$), and neglecting the exponentially decreasing corrections we find

$$F_q^1(R,L) + F_q^b(R,L) = -\frac{\pi L}{24R} \left(1 + 2\frac{b}{R} \right)^{\frac{1}{2}}, \tag{5.4}$$

which becomes at the first order in b

$$F_q^1(R,L) + F_q^b(R,L) = -\frac{\pi L}{24R} \left(1 + \frac{b}{R} \right), \tag{5.5}$$

which is exactly Lüscher and Weisz's result [2].

5.2 Large distance behaviour of the boundary correction

The major advantage of having the complete functional form of the boundary correction is that we can now look at its large distance behaviour. By using the modular transformation of the Dedekind function we find for $2R^* > L$

$$F_q^1(R,L) + F_q^b(R,L) = \left[-\frac{\pi R^*}{6L} + \frac{1}{2} \log \frac{2R^*}{L} + \sum_{n=1}^{\infty} \log(1 - e^{-4\pi n R^*/L}) \right]. \tag{5.6}$$

Neglecting again the exponentially decreasing terms, and keeping only the first order in b we end up with

$$F_q^1(R,L) + F_q^b(R,L) = \left[-\frac{\pi R}{6L} \left(1 - \frac{b}{R} \right) + \frac{1}{2} \log \frac{2R}{L} - \frac{1}{4} \log \left(1 + 2\frac{b}{R} \right) \right], \tag{5.7}$$

which means (remember again that we only keep the terms proportional to b in the expansion)

$$F_q^1(R,L) + F_q^b(R,L) = -\frac{\pi R}{6L} + \frac{\pi b}{6L} + \frac{1}{2} \log \frac{2R}{L} - \frac{b}{2R} \tag{5.8}$$

from which we obtain

$$F_q^b(R,L) = +\frac{\pi b}{6L} - \frac{b}{2R}\,. \tag{5.9}$$

Thus we see that the effect of the boundary term is to renormalize the constant term $k(L)$ (which however is not relevant in the ratios evaluated in Monte Carlo simulations) and to add a further $1/R$ correction.

A few observations are in order at this point.

- This result is rather satisfactory from a physical point of view: the effect of a boundary term can only decrease as a function of R (i.e. when the two boundaries are far apart), and this is indeed the case in the large R regime, too. This is in sharp contrast with the behaviour of the string fluctuations, whose dominant term in the large distance regime is always (i.e. at any order in the $1/\sigma RL$ expansion) a linearly rising correction.
- Although both in the short distance and in the large distance regimes the correction induced by the boundary term has the same sign, nevertheless there is a well defined change of behaviour: from $-b\pi L/24R^2$ to $-b/2R$. This feature may simplify the identification of such a term in numerical simulations. Notice in particular the lack of L dependence in the large R regime.

6. Comparison with the Ising gauge model in the large distance regime

The aim of this section is to compare the prediction of eq. (5.8) with the results of the simulations reported in [4] for the large distance regime of the interquark potential in the 3d gauge Ising model. In particular, we shall concentrate on the set of simulations performed at $\beta = 0.75180$, which were the most precise ones. The simulations were performed on lattices with space-like size $N_s = 128$ and "time" sizes $L = 10, 12, 16, 24$, corresponding to $\frac{4}{5}T_c$, $\frac{2}{3}T_c$, $\frac{1}{2}T_c$ and $\frac{1}{3}T_c$, respectively. For each temperature, we studied the correlators for various values of the interquark distance R, ranging from $R = 8$ to $R = 48$. The data are reported in [4].

The most effective observable to identify the possible presence of a boundary correction is the following combination

$$\begin{aligned} Q_2(R) &\equiv -\log\frac{G(R+1)}{G(R)} - F_q^1(R+1) + F_q^1(R) \\ &= -\log\frac{G(R+1)}{G(R)} + \log\left[\eta\left(i\frac{L}{2R}\right)\right] - \log\left[\eta\left(i\frac{L}{2(R+1)}\right)\right] \quad . \end{aligned} \tag{6.1}$$

In order to understand the results of the fit it may be useful to extract from eqs (2.4) and (2.8) the expected behaviour in the large distance regime for this quantity (which as we mentioned several times is drastically different from the short distance one discussed in sections 3 and 4). Discarding exponentially decreasing terms, we find

$$F_q^1(R+1,L) - F_q^1(R,L) = -\frac{\pi}{6L} + \frac{1}{2}\log\left(\frac{R+1}{R}\right) \tag{6.2}$$

L	χ^2_r	b_0	$1/8\sigma L$
10	2.72	-1.31(7)	-1.184
12	2.33	-0.97(6)	-0.986
16	2.30	-0.62(5)	-0.740
24	2.54	-0.30(4)	-0.493

Table 12: Results of the fits to eq. (6.5). In the first column the value of L, in the second column the reduced χ^2 is given, in the last two columns the values of b_0 and the corresponding expectation — see eq. (6.3) — according to the Nambu-Goto string.

and (see eqs (2.37) of ref. [4])

$$F_q^2(R+1,L) - F_q^2(R,L) = -\frac{\pi^2}{72\sigma L^3} - \frac{1}{8\sigma L}\frac{1}{R(R+1)}\,. \tag{6.3}$$

These corrections should be compared with what is expected from the boundary term, namely

$$F_q^b(R+1,L) - F_q^b(R,L) = \frac{b}{2R(R+1)}\,. \tag{6.4}$$

The boundary correction to $Q_2(R)$ has the same R dependence as the Nambu-Goto term, but a *different L dependence* (this is an obvious consequence of the fact that b is a dimensional parameter): that makes it very easy to disentangle between the two corrections. In particular, fitting them with the data of [4], we find that b is compatible with zero, i.e. that no boundary correction seems to be present in the 3d Ising gauge model.

More precisely, we fitted $Q_2(R)$, for each of the four values of L, with the following formula

$$Q_2(R) = a_0 + b_0\frac{1}{R(R+1)}\,, \tag{6.5}$$

where a_0 is related to the string tension, while b_0 encodes the large distance $1/R$ correction to the potential we are interested in. With this choice of normalization, if the $1/R$ correction is completely due to the boundary term, then $b = 2b_0$. Since for this value of β we have $R_c \sim 12$, we only fitted the data for $R \geq 16$, for a total of 6 points. The results of the fits are reported in table 12.

Few observation can be made on these results

(*a*) The magnitude of the reduced χ^2 (which is remarkably constant as L changes) indicates that in the large distance regime the Nambu-Goto string (truncated at the first perturbative order) is a reasonable approximation, but does not fully describe the data.

(*b*) There is a clear L dependence in the best fit values of b_0. These values qualitatively agree with the Nambu-Goto expectation, but they seem to show an even steeper dependence on L. Apparently little room is left for a boundary correction term. A naive extrapolation to $L \to \infty$ suggests a very small value of b, most probably compatible with zero.

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(c) It is interesting to compare what we find here with the fits discussed in section 4 (see tables 9, 10 and 11) we find for this value of β: $b_0 \equiv b/2 = -\frac{12}{\pi}\frac{\gamma_2}{\sqrt{\sigma}} \sim -0.35$ which is of the same order of magnitude and has the same sign of the values reported in table 12.

7. Concluding remarks

Let us summarize the main results of our analysis.

(a) We obtained an explicit form for the boundary correction, at first perturbative order in b, which is valid for any value of R and L. The importance of this result is that in the large R regime, by virtue of the peculiar form of the boundary correction, i.e. the fact that it is L-independent, it is much simpler to measure the value of b.

(b) As a first application of this result, we evaluated b in the 3d gauge Ising model, using the large distance data published in [4]. It turns out that b is very small and compatible with zero.

(c) In order to test the effective string picture in the short distance regime as well, we performed a new set of simulations for the 3d gauge Ising model. A careful analysis of the data in the range of T and R values in which higher order terms can be neglected shows a perfect agreement with the prediction of the free bosonic string. However it turns out that in order to describe the data one must keep into account *the whole functional form of the correction*. Cutting it to the Lüscher term only may induce a relevant mismatch with the data.

(d) Finally we studied the higher order corrections for small values of R. In this regime the Nambu-Goto correction, truncated at the first perturbative order, should behave as $1/R^3$. Assuming $b = 0$ and fitting the data with this correction (the $\alpha = 3$ fit in the section 4) we found an acceptable χ^2_r, but the wrong coefficient.

This is indeed rather a puzzling result since in the large distance, high temperature regime we found instead quite a good agreement with the Nambu-Goto prediction (see [4]). In principle, a description holding at short distances could even be a picture different from an effective string [9]. However, it is also possible that the short distance breakdown of the Nambu-Goto string scenario could be explained within the framework of an effective string theory. In particular, following the comments that we made at the end of section 4, we see two possible explanations for this behaviour.

1. The Nambu-Goto picture could be the wrong assumption. In this framework the good agreement at large distance is only a coincidence, and one should look for some other, more exotic, effective string action to simultaneously describe the short distance as well as the finite temperature behaviour of the potential. Notice however that none of the generalized strings discussed in [15] satisfies these constraints, thus some drastically different model would be needed to follow this line.

2. The short distance deviation with respect to the Nambu-Goto prediction might be due to an irrelevant operator which may possibly have a string-like description. Two natural options are:

    - a higher order boundary term which decreases with a higher power of R and could thus be compatible both with the behaviour that we extract from the short distance fits (which suggests a power higher than $1/R^2$) and with the large distance regime.
    - a term proportional to the extrinsic curvature of the string [20, 21]. It is possible that such a term gives significantly different contributions in presence of Dirichlet and periodic boundary conditions.

    In this framework it could be useful to compare the short distance behaviour of different models to see if some common feature emerges. The nice feature of this last option is that it does not require a large value of the b parameter and thus it does agree with all our numerical results.

Following the suggestion of the last point above we tried to test if some common behaviour is shared by different gauge theories. To this end, we performed the same analysis discussed in section 4 with the data on the 3d SU(3) gauge model published in [2]. In particular, we concentrated on the sample at $\beta = 20$, see the data in table 3 of ref. [2], which correspond to a value $\sigma \sim 0.0342$. In agreement with the results of [2], we found that the data are well described by the free string correction only, with apparently no need of higher order corrections. The most impressive signature of this behaviour is given by the observable $D(R)$ introduced in section 4 which is different from zero only if higher order corrections are present. It turns out that for SU(3), $D(R)$ is almost compatible with zero within the errors, for all of the values $\sqrt{\sigma}R > 1.3$. To complete the comparison we also studied a set of data obtained in the 3d SU(2) model taken from [22]. These data are characterized by a slightly smaller string tension $\sigma \sim 0.0263$ and a temperature $T/T_c \sim 1/7$ (see [22] for details). The value of $D(R)$ for these two sets of data, together with the $L = 80$, $\beta = 0.73107$ sample of the Ising model are reported in figures 6 and 7. Notice that the SU(3) and SU(2) samples have values of σ which are smaller than the Ising one. Looking at these figures one sees that the SU(2) data lie somehow in between the Ising and the SU(3) ones. This is also confirmed by the γ_3 value extracted from the SU(2) data: $\gamma_3 \sim 0.004$ [22] which is smaller (and outside the error bars) than the one that we obtained in the present paper for the Ising case.

A similar comparison between the 3D SU(2) and $\mathbb{Z}_2$ gauge models for the first excited string level can be found in ref. [9]. In this case, the SU(2) and $\mathbb{Z}_2$ data seem to be compatible.

Further studies are needed to better characterize these behaviours, to understand if they may be related to some feature of the gauge group (say, for instance, to the center of the group as it is suggested by the analogy between Ising and SU(2) behaviours) and to see if they support any of the scenarios proposed above.

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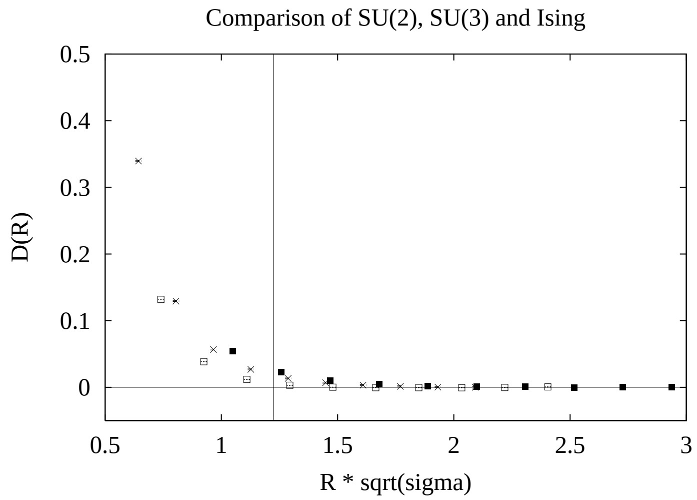

Figure 6: Values of $D(R)$ versus $R\sqrt{\sigma}$ for the $d = 3$ SU(3) gauge theory (white squares), the Ising gauge model (black squares) and the 3d SU(2) gauge theory (crosses). The vertical line corresponds to $R = R_c$.

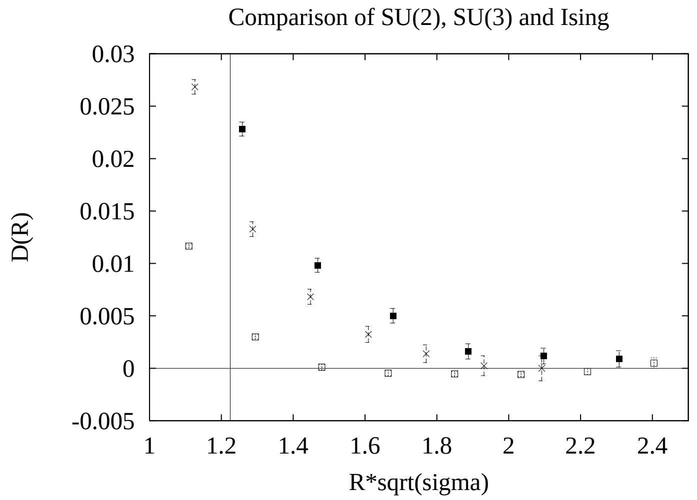

Figure 7: Same as figure 6 but with a higher resolution.

Acknowledgments

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A. Evaluation of the functional form of the boundary correction

Let us begin by considering the case when we assume that the dynamics of the effective string world sheet is described by a pure Nambu-Goto action, neglecting a possible boundary term: under this hypothesis, the action is given by the string world sheet surface area

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(measured in units of the inverse string tension), and the expectation value for the Polyakov loop correlator can be expressed as the partition function:

$$Z = \int [\mathcal{D}h] \, e^{-\sigma \mathcal{A}_0} \, . \tag{A.1}$$

The Nambu-Goto action $\mathcal{A}_0$ can be written in the following form:

$$\mathcal{A}_0 = \int_0^R dz \int_0^L dt \sqrt{\det g_{\alpha\beta}} \tag{A.2}$$

where $g_{\alpha\beta}$ is defined as:

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \, . \tag{A.3}$$

Let us focus on the case of three dimensions: we can parametrize the X field in the following way:

$$\begin{aligned} X_1 &= z \, ; \\ X_2 &= t \, ; \\ X_3 &= h = h\,(z,t) \, ; \end{aligned} \tag{A.4}$$

and the h field is associated to the string world sheet surface transverse displacement with respect to a minimal area plane. Thus we have:

$$\mathcal{A}_0 = \int_0^R dz \int_0^L dt \sqrt{1 + \left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2} \, . \tag{A.5}$$

A Taylor expansion of the square root appearing in Equation (A.5) gives — to first non-trivial order:

$$\mathcal{A}_0 \simeq \int_0^R dz \int_0^L dt \left\{ 1 + \frac{1}{2} \left[\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2 \right] \right\} \tag{A.6}$$

and the partition function can be approximated as:

$$Z_{0+1} = e^{-\sigma R L} \int [\mathcal{D}h] \, e^{-\frac{\sigma}{2} \int_0^R dz \int_0^L dt \left[\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2 \right]} \, . \tag{A.7}$$

On the other hand, if we allow for the existence of a possible "boundary term", the effective action will take the form:

$$Z^\star = \int [\mathcal{D}h] \, e^{-\sigma \mathcal{A}} \tag{A.8}$$

with:

$$\mathcal{A} = \mathcal{A}_0 + \frac{b}{4} \int_0^L dt \left[\left(\frac{\partial h}{\partial z}\right)^2_{z=0} + \left(\frac{\partial h}{\partial z}\right)^2_{z=R} \right] \tag{A.9}$$

where b is a parameter with dimensions [length], while the $\frac{1}{4}$ factor is just a conventional choice (see also [2]).

Let us notice that, in the case of the system we are presently interested in, namely: a finite temperature confining gauge theory,[4] the z-dependence of the eigenfunctions appropriate to the case of the pure Nambu-Goto action can be factored out as:

$$\sin\left(\frac{n\pi z}{R}\right), \qquad n \in \mathbb{N}_0 \tag{A.10}$$

thus:

$$\frac{\partial h}{\partial z} = \frac{n\pi}{R}\cos\left(\frac{n\pi z}{R}\right)\dots \tag{A.11}$$

where we omitted the eigenfunction term expressing the t-dependence; thus we have:

$$\begin{aligned}\left(\frac{\partial h}{\partial z}\right)^2_{z=0} = \left(\frac{\partial h}{\partial z}\right)^2_{z=R} &= \frac{n^2\pi^2}{R^2}\cos^2\left(\frac{n\pi z}{R}\right)\Big|_{z=0}\dots = \frac{n^2\pi^2}{R^2}\cos^2\left(\frac{n\pi z}{R}\right)\Big|_{z=R}\dots \\ &= \frac{n^2\pi^2}{R^2}\dots\,. \end{aligned}\tag{A.12}$$

As it is concerned with the momentum in the compactified direction, the eigenvalues are:

$$\frac{4\pi^2 m^2}{L^2}, \qquad m \in \mathbb{Z}\,. \tag{A.13}$$

If we define:

$$\begin{cases}\rho = R \\ \tau \;\; = \dfrac{L}{2}\end{cases} \tag{A.14}$$

the eigenvalues of the $(-\Delta)$ operator can be written as:

$$\pi^2\left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right). \tag{A.15}$$

Taking into account the boundary term too, the complete action $\mathcal{A}$ reads, to first non-trivial order:

$$\mathcal{A} \simeq \int_0^R dz \int_0^L dt \left\{1 + \frac{1}{2}\left[\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2\right]\right\} + \frac{b}{4}\int_0^L dt\left[\left(\frac{\partial h}{\partial z}\right)^2_{z=0} + \left(\frac{\partial h}{\partial z}\right)^2_{z=R}\right]. \tag{A.16}$$

Now, we propose to work out the correction to the $V(R)$ interquark potential to first order in b in a perturbative expansion; to do this, we have to evaluate:

$$Z^{\star}_{0+1} = \left[\int [\mathcal{D}h]\, e^{-\sigma\mathcal{A}}\right]_{\text{up to first order}}. \tag{A.17}$$

The Taylor expansion of the exponent gives:

$$\sigma\mathcal{A} \simeq \sigma RL + \frac{\sigma}{2}\int_0^R dz\int_0^L dt\left[\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2\right] + \frac{\sigma b}{4}\int_0^L dt\left[\left(\frac{\partial h}{\partial z}\right)^2_{z=0} + \left(\frac{\partial h}{\partial z}\right)^2_{z=R}\right]. \tag{A.18}$$

[4]The string world sheet associated to the expectation value of the Polyakov loop correlator in a finite temperature gauge theory is characterized by fixed boundary conditions in the z direction, and periodic boundary conditions in the compactified, t direction.

We can think about this formula as the starting point in a perturbative analysis in terms of the b parameter, namely we can consider the boundary term as a perturbation of the pure Nambu-Goto action eq. (A.2).

First of all, let us notice that:

1. by virtue of the fact that the eigenvalues of the unperturbed problem are known — see eq. (A.15) — the double integral appearing in eq. (A.18) can be written in a simpler way:

$$\int_0^R dz \int_0^L dt \left[\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial t}\right)^2 \right] = \pi^2 \left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right) \cdot \int_0^R dz \int_0^L dt \, (\ldots)_{[z,t]^2} \quad (A.19)$$

   where the notation $(\ldots)_{[z,t]^2}$ represents (the square modulus of) the complete, unperturbed eigenfunction.

2. On the other hand, eq. (A.12) shows that the two addends appearing in the integral associated to the boundary term in eq. (A.18) are equal, and they give rise to a contribution which can be written as:

$$\int_0^L dt \left[\left(\frac{\partial h}{\partial z}\right)^2_{z=0} + \left(\frac{\partial h}{\partial z}\right)^2_{z=R} \right] = 2 \frac{n^2 \pi^2}{R^2} \cdot \int_0^L dt \, (\ldots)_{[t]^2} \, . \quad (A.20)$$

   In this case, the integrand $(\ldots)_{[t]^2}$ is just (the square modulus of) the t-dependent part of the eigenfunction.[5]

Thus, in the "pure Nambu-Goto" case, $\sigma \mathcal{A}_0$ involves terms like:

$$\frac{\sigma}{2}\pi^2 \left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right) \cdot \int_0^R dz \int_0^L dt \, (\ldots)_{[z,t]^2} \quad (A.21)$$

but when the boundary term is included this expression has to be replaced by:

$$\frac{\sigma}{2}\pi^2 \left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right) \cdot \int_0^R dz \int_0^L dt \, (\ldots)_{[z,t]^2} + \frac{\sigma b}{2} \frac{n^2 \pi^2}{R^2} \cdot \int_0^L dt \, (\ldots)_{[t]^2} \quad (A.22)$$

The integrals appearing in eq. (A.22) give rise to results[6] in the form:

$$\int_0^R dz \int_0^L dt \, (\ldots)_{[z,t]^2} = C_2 \cdot RL \quad (A.23)$$

$$\int_0^L dt \, (\ldots)_{[t]^2} = C_1 \cdot L \quad (A.24)$$

where C_2 and C_1 are just pure numbers;[7] thus we can rewrite eq. (A.22) as:

$$\sigma \frac{\pi^2}{2} \left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right) \cdot C_2 RL + \frac{\sigma b}{2} \pi^2 \frac{n^2}{R^2} \cdot C_1 L = \sigma RL \frac{\pi^2}{2} C_2 \cdot \left[\left(\frac{n^2}{\rho^2} + \frac{m^2}{\tau^2}\right) + \frac{b}{R} \cdot \frac{C_1}{C_2} \cdot \frac{n^2}{\rho^2} \right] . \quad (A.25)$$

[5] The z-dependence does not appear, because we already evaluated its value in $z = 0$ and in $z = R$.

[6] The results appearing in eq. (A.23) and eq. (A.24) refer to an unnormalized eigenfunction, but this is not relevant to our present calculation, since the only difference in the normalized eigenfunction case is that both quantities are multiplied by a common factor, and their ratio is unchanged.

[7] Strictly speaking, this is true only if we do not consider the usual normalization factor; however, even in the more general case, the ratio between C_2 and C_1 will be the same as we find here.

Defining:

$$k = \frac{C_1}{C_2} \tag{A.26}$$

then eq. (A.25) can be rewritten as:

$$\sigma\frac{\pi^2}{2}\cdot C_2RL\left[\frac{m^2}{\tau^2}+\frac{n^2}{R^2}\left(1+k\frac{b}{R}\right)\right] = \sigma\frac{\pi^2}{2}\cdot C_2RL\left\{\frac{m^2}{\tau^2}+\frac{n^2}{\left[R(1+k\frac{b}{R})^{-\frac{1}{2}}\right]^2}\right\}. \tag{A.27}$$

By defining:

$$R^\star = R\left(1+k\frac{b}{R}\right)^{-\frac{1}{2}} \tag{A.28}$$

eq. (A.27) takes the form:

$$\sigma\frac{\pi^2}{2}\cdot C_2RL\left(\frac{m^2}{\tau^2}+\frac{n^2}{R^{\star 2}}\right). \tag{A.29}$$

In the pure Nambu-Goto theory (with no boundary term), the corresponding term:

$$\sigma\frac{\pi^2}{2}\cdot C_2RL\left(\frac{m^2}{\tau^2}+\frac{n^2}{R^2}\right) \tag{A.30}$$

is associated to the Z_1 term in the partition function factorization formula eq. (A.7); in three dimensions, Z_1 turns out to be equal to:

$$Z_1 = \left[\eta\left(i\frac{\tau}{\rho}\right)\right]^{-1} = \left[\eta\left(i\frac{L}{2R}\right)\right]^{-1} \tag{A.31}$$

and its (additive) contribution to the interquark potential is:

$$V_1(R) = -\frac{1}{L}\log Z_1 = \frac{1}{L}\log\eta\left(i\frac{L}{2R}\right). \tag{A.32}$$

Since Dedekind's function is defined as:

$$\eta\left(i\frac{\tau}{\rho}\right) = q^{\frac{1}{24}}\prod_{n=1}^{+\infty}(1-q^n)\,, \qquad q = e^{-2\pi\frac{\tau}{\rho}}\,. \tag{A.33}$$

In the regime $L/2R \gg 1$, we can approximate:

$$\begin{aligned} V_1(R) &= \frac{1}{L}\left[-\frac{\pi L}{24R}+\sum_{n=1}^{+\infty}\log\left(1-e^{-\frac{\pi L}{R}n}\right)\right] \\ &\simeq -\frac{\pi}{24R} \end{aligned} \tag{A.34}$$

which is nothing but the celebrated "Lüscher term" [12].

When the boundary term is included in the action, we get a similar result, provided we replace:

$$R \longrightarrow R^\star = R\left(1+k\frac{b}{R}\right)^{-\frac{1}{2}}. \tag{A.35}$$

We can evaluate k in simple way: its definition is given by eq. (A.26), which involves the integrals in eq. (A.23) and in eq. (A.24), thus:

$$k = \frac{C_1}{C_2} = R \cdot \frac{\int_0^L dt \, (\dots)_{[t]^2}}{\int_0^R dz \int_0^L dt \, (\dots)_{[z,t]^2}}. \tag{A.36}$$

Since the two eigenfunction parts expressing the t-dependence and z-dependence are factorized, the denominator integrals factorize, too. Moreover, we know that the z-dependent part of the eigenfunction is expressed by eq. (A.10); these observations allow one to write:

$$k = R \frac{\int_0^L dt \, (\dots)_{[t]^2}}{\int_0^R \sin^2\left(\frac{n\pi z}{R}\right) \int_0^L dt \, (\dots)_{[t]^2}} = R \cdot \frac{1}{\frac{1}{2} R} = 2 \tag{A.37}$$

where we exploited the fact that $n \in \mathbb{N}_0$.

Thus, in the case when we allow for the presence of the boundary term, the first-order non-trivial contribution to the interquark potential will be:

$$V_1^\star = \frac{1}{L} \log \eta \left(i \frac{L}{2R^\star} \right). \tag{A.38}$$

It is important to stress the fact that this result does not rely on the assumption to be in the "large $L/2R$ regime": as a matter of fact, the only hypothesis for the validity of eq. (A.35) is that b can be considered as a perturbatively small parameter.

On the other hand, if we restrict to the case when $L/2R \gg 1$, then eq. (A.38) can be approximated by:

$$\begin{aligned} V_1^\star &\simeq -\frac{\pi}{24R^\star} = -\frac{\pi}{24R\left(1+2\frac{b}{R}\right)^{-\frac{1}{2}}} \\ &= -\frac{\pi}{24R}\sqrt{1+2\frac{b}{R}} \simeq -\frac{\pi}{24R}\left(1+\frac{b}{R}\right) \end{aligned} \tag{A.39}$$

which exactly reproduces the term appearing in eq. (3.9) in [2].

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